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**LEUVEN**

Faculty of Economics and  
Applied Economics

Department of Economics

Bounded Sequential Dominance Criteria

by

Erwin OOGHE

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**DISCUSSION  
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# Bounded sequential dominance criteria

Erwin Ooghe\*

CES, Katholieke Universiteit Leuven

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## Abstract

In the spirit of Fleurbaey et al. (2001), it is tempting to introduce more reasonable lower *and* upper bounds in Atkinson and Bourguignon's (1987) sequential generalized Lorenz dominance procedure. Unfortunately, our proposal leads, at best, to an average household income criterion, which is unsuited to make heterogeneous welfare comparisons. We therefore restrict attention to imposing either lower or upper bounds, resulting in two new sequential dominance criteria.

## 1 Introduction

When income units are homogeneous in non-income characteristics, there exist many tools to make social welfare comparisons between distributions and the properties of these tools are well-known (Lambert, 2001 for an overview). Especially, the generalized Lorenz dominance test seems well-accepted as a powerful tool to measure social welfare. Unfortunately, these tools are too restrictive to make reasonable comparisons in practice, because:

At the heart of any distributional analysis, there is the problem of allowing for differences in people's non-income characteristics. (Cowell and Mercader-Prats, 1999)

If we do not want to cardinalize need differences via equivalence scales (Shorrocks, 1995, Ebert, 1997), the most well-known way to make heterogeneous welfare comparisons is the so-called "ordinal" sequential generalized Lorenz dominance test. It boils down to classifying households in different need types and checking —on the basis of the generalized Lorenz dominance criterion— (i) whether the most needy are better off, (ii) whether the most and second most needy are better off, and so on. This result is due to Atkinson and Bourguignon (1987), and extended by Atkinson (1992), Jenkins and Lambert (1993), Chambaz and Maurin (1998), Lambert and Ramos (2002), and Moyes (1999) to deal with changing demographics, poverty and/or the principle of diminishing transfers. Related to this literature is Bourguignon's (1989) dominance criterion —which is a subrelation of (and thus more incomplete compared to) the sequential generalized Lorenz dominance quasi-ordering.

The main criticism of the ordinal approach is its impossibility to assess, for example, an income transfer from a very rich household to a poor household with lower needs.

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\*Postdoctoral Fellow of the Fund for Scientific Research - Flanders. Center for Economic Studies, Naamsestraat, 69, B-3000 Leuven, Belgium. e-mail to [erwin.ooghe@econ.kuleuven.ac.be](mailto:erwin.ooghe@econ.kuleuven.ac.be).

Recently, Fleurbaey et al. (2001) provide an intermediate criterion, which at the same (i) deals with this criticism in a convincing way by introducing “bounded” household utility functions, and (ii) bridges the gap between the cardinal and the ordinal approach. Their criterion is equivalent with Ebert’s (1997,1999) cardinal approach for a set of bounded equivalence scales. Choosing these bounds as small (resp. as wide) as possible, one obtains Ebert’s cardinal approach (resp. Bourguignon’s “ordinal” approach).

*First*, we analyze whether it is also possible to consider household utility functions, bounded from below and above, in Atkinson and Bourguignon’s “ordinal” approach. The answer is rather dissapointing: the resulting criterion focusses on average household income, which is of course too restrictive to make heterogeneous welfare comparisons.

*Second*, we look what happens when we impose either lower or upper bounds, rather than both simultaneously. We obtain two new sequential dominance criteria, which are basically sequential procedures applied to household incomes divided and weighted by an equivalence scale, which is based on either the lower or the upper bounds. The first one (based on lower bounds) is able to deal with transfers from rich households to poor and more needy households, the second one (based on upper bounds) is able to deal with transfers from rich households to poor and less needy households.

## 2 Notation

Consider household incomes  $y \in \mathbb{R}_+$  and types  $k \in \mathbb{K} = \{1, \dots, K\}$  representing relevant non-income characteristics; types are ordered from most to least needy. A heterogeneous distribution  $F$  is a list  $(p_1, \dots, p_K, F_1, \dots, F_K)$ , with  $p_k$  the proportion of households with type  $k$  and  $F_k$  the income distribution function of type  $k$  households, assumed to be continuously differentiable and defined over a common finite support  $[0, z]$  (and thus equal to one outside this support). We focus directly on the case where demographics might change, or the proportions  $p_k$  may differ between heterogeneous distributions. Finally, we introduce (twice differentiable) household utility functions  $U_k : \mathbb{R}_+ \rightarrow \mathbb{R}$ , measuring the utility of a household with type  $k$  as a function of its income. Social welfare in a distribution  $F$  is measured by the average household utility in society:

$$W : F \mapsto W(F) = \sum_{k \in \mathbb{K}} p_k \int_0^z U_k dF_k. \quad (1)$$

## 3 Welfare properties

In the first subsection, we present Atkinson and Bourguignon’s (1987) sequential generalized Lorenz dominance criterion, as extended by Moyes (1999) to deal with changing demographics. In a second subsection, we introduce bounds in the spirit of Fleurbaey et al. (2001).

### 3.1 Sequential dominance

We might impose one or more of the following conditions on utility profiles  $(U_1, \dots, U_K)$  (an explanation follows):

- **A1:**  $U'_k \geq 0$ , for all  $k \in \mathbb{K}$  (A1a) and  $U''_k \leq 0$ , for all  $k \in \mathbb{K}$  (A1b).

- **A2:**  $U'_k(y) - U'_{k+1}(y) \geq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K - 1$ .
- **A3:**  $\left( U'_k(y) - U'_{k+1}(y) \right)' \leq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K - 1$ .
- **A4:**  $U_k(y) - U_{k+1}(y) \leq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K - 1$ .

Assumption A1 is standard: the marginal utility of income —called social priority in the sequel, as it tells us where to put our money first when maximizing the sum of utilities— of all household types is positive, but decreases with income. Assumption A2 requires the social priority to be higher for more needy households, but A3 says that the difference in social priority between adjacent types decreases with income. Finally, A4 is a condition to deal with changing demographics: more needy households are worse off compared to less needy households with the same household income. Bourguignon (1989) considers assumptions A1-A2, Atkinson and Bourguignon (1987) A1-A3 and Moyes (1999) A1-A4.

Following Ebert (2000), we can also interpret conditions A1-A3 in terms of money transfers between households. A1a —known as the Pareto condition— ensures that more income improves social welfare, whereas A1b —known as the Pigou-Dalton transfer principle— tells us that an income transfer from a rich to a poor household with the same type increases welfare. Adding A2, requires that transferring money from a rich to a poor household increases social welfare, as long as the poor household is more needy. Adding A3, ensures that transferring money from a rich to a poor household with the same type would increase social welfare more, when these households are more needy. We denote with  $\mathcal{U}$  the family of utility profiles  $(U_1, \dots, U_K)$  satisfying assumptions A1-A4. We say that a distribution  $F$  welfare dominates  $G$  according to the family  $\mathcal{U}$ , denoted  $F \succsim G$  if and only if the welfare difference  $\Delta W = W(F) - W(G)$  is positive for all profiles in  $\mathcal{U}$ . The following theorem shows how welfare dominance can be implemented via sequential dominance conditions. Define functions  $H_k^1, H_k^2$  on  $\mathbb{R}_+$ , for all  $k \in \mathbb{K}$ , as

$$H_k^1(y) = p_k F_k(y) - q_k G_k(y), \text{ and } H_k^2(y) = \int_0^y H_k^1. \quad (2)$$

We obtain (an explanation follows):

**Theorem 1.** Consider two heterogeneous distributions  $F$  and  $G$  defined over a common finite support  $[0, z]$ . We have  $F \succsim G \Leftrightarrow$

$$\sum_{i=1}^k H_i^1(z) \leq 0 \text{ for all } k \in \mathbb{K}, \text{ and} \quad (3)$$

$$\sum_{i=1}^k H_i^2(y) \leq 0 \text{ for all } y \text{ in } [0, z] \text{ and for all } k \in \mathbb{K}. \quad (4)$$

**Proof.** See Moyes (1999); see also the proof of proposition 2, as it contains theorem 1 as a special case.  $\square$

According to the conditions in (3), we have to check first whether the proportion of most needy households is smaller in  $F$  compared to  $G$ , second, whether the proportion of most and second most needy households is smaller in  $F$  compared to  $G$ , and so on.

The conditions in (4) test for second-order stochastic dominance in the same sequential way. Notice that the test in theorem 1 reduces to the sequential generalized Lorenz dominance test, whenever demographics are the same in both distributions.

### 3.2 Introducing bounds

The traditional sequential dominance criterion, described in the previous subsection, cannot deal with transfers from rich households to poor but less needier households. The main problem is assumption A2, as it allows for utility profiles where, for example, a couple with ten times the income of a single has a higher social priority. Fleurbaey et al.'s (2001) contribution consists of introducing more reasonable lower and upper bounds.

Lower and upper bound vectors will be denoted by  $\alpha = (\alpha_1, \dots, \alpha_K)$  and  $\beta = (\beta_1, \dots, \beta_K)$ ; they satisfy  $\beta_k \geq \alpha_k \geq 1$  for all  $k \in \mathbb{K}$  and for the reference type, type  $K$ , we choose  $\beta_K = \alpha_K = 1$ . Fleurbaey et al. (2001) replace Bourguignon's (1989) assumption A2 by the bounded assumptions  $A2_\alpha$  and  $A2^\beta$  (an explanation follows):

- **A2 $_\alpha$**  :  $U'_k(\alpha_k y) - U'_{k+1}(y) \geq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K-1$ .
- **A2 $^\beta$**  :  $U'_k(\beta_k y) - U'_{k+1}(y) \leq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K-1$ .

A2 $_\alpha$  (resp. A2 $^\beta$ ) tells us that a household with type  $k$  has a higher (resp. lower) social priority compared to a household with type  $k+1$ , if the former's household income is sufficiently low (resp. high), i.e., lower (resp. higher)<sup>1</sup> or equal to  $\alpha_k$  (resp.  $\beta_k$ ) times the latter's household income.

To extend Fleurbaey et al.'s (2001) analysis into the direction of Atkinson and Bourguignon (1987), it seems natural to replace assumption A3 by assumptions A3 $_\alpha$  and A3 $^\beta$ , which require that the (positive) differences in social priority, described in assumptions A2 $_\alpha$  and A2 $^\beta$ , decrease with income:

- **A3 $_\alpha$**  :  $\left( U'_k(\alpha_k y) - U'_{k+1}(y) \right)' \leq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K-1$ .
- **A3 $^\beta$**  :  $\left( U'_k(\beta_k y) - U'_{k+1}(y) \right)' \geq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K-1$ .

Finally, to deal with changing demographics, we replace A4 by assumptions A4 $_\alpha$  and A4 $^\beta$  (an explanation follows):

- **A4 $_\alpha$**  :  $U_k(\alpha_k y) - U_{k+1}(y) \leq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K-1$ .
- **A4 $^\beta$**  :  $U_k(\beta_k y) - U_{k+1}(y) \geq 0$ , for all  $y \in \mathbb{R}_+$ , for all  $k = 1, \dots, K-1$ .

Assumption A4 $_\alpha$  (resp. A4 $^\beta$ ) tells us that a household with type  $k$  is worse off (resp. better off) compared to a household with type  $k+1$ , if the former's household income is sufficiently low (resp. high), i.e., lower (resp. higher) or equal to  $\alpha_k$  (resp.  $\beta_k$ ) times the latter's household income.

In the next section, we focus on the following quasi-orderings:

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<sup>1</sup>For this interpretation to be correct, we need A1 to be satisfied.

$\succsim_\alpha^\gamma$  : welfare dominance based on  $\mathcal{U}_\alpha = \{(U_1, \dots, U_K) \text{ satisfying A1, A2}_\alpha, \text{A3}_\alpha, \text{A4}_\alpha\}$ ,  
 $\succsim_\alpha^\beta$  : welfare dominance based on  $\mathcal{U}^\beta = \{(U_1, \dots, U_K) \text{ satisfying A1, A2}^\beta, \text{A3}^\beta, \text{A4}^\beta\}$ ,  
 $\succsim_\alpha^\beta$  : welfare dominance based on  $\mathcal{U}_\alpha^\beta = \mathcal{U}_\alpha \cap \mathcal{U}^\beta$ .

## 4 Results

Our first result tells us that our attempt to find an intermediate criterion in between Ebert's cardinal and Atkinson and Bourguignon's ordinal approach is unsatisfactory, as it results, at best, in an average household income criterion. Notice that (i) assumptions A4 $_\alpha$  and A4 $^\beta$  do not play a role in proposition 1 and (ii) proposition 1 does not hold when  $\alpha = \beta$ , as it would lead to Ebert's cardinal approach.

**Proposition 1.** The household utility functions in a profile  $(U_1, \dots, U_K)$  satisfying A1, A2 $_\alpha$ , A2 $^\beta$ , A3 $_\alpha$ , and A3 $^\beta$  for some lower and upper bound vectors

$$\beta = (\beta_1, \dots, \beta_{K-1}, 1) > \alpha = (\alpha_1, \dots, \alpha_{K-1}, 1) \geq (1, \dots, 1),$$

have the following form:

$$U_k : \mathbb{R}_+ \rightarrow \mathbb{R} : y \mapsto a_k + by, \text{ with } a_k \in \mathbb{R} \text{ and } b \in \mathbb{R}_+.$$

**Proof.** See appendix. □

As a consequence, the criterion  $\succsim_\alpha^\beta$  cannot make reasonable heterogeneous welfare comparisons. In the sequel, we look at the quasi-orderings  $\succsim_\alpha$  and  $\succsim^\beta$ , which are based either on lower or upper bounds.

We need two new definitions. For all  $k \in \mathbb{K}$ , define the lower and upper bound equivalence scale with respect to type  $K$  (the reference type) as  $\alpha_k^* = \prod_{i=k}^K \alpha_i$  and  $\beta_k^* = \prod_{i=k}^K \beta_i$ ;  $\alpha^*$  and  $\beta^*$  are the corresponding vectors. Define functions  $\underline{H}_k^2$  and  $\overline{H}_k^2$  on  $\mathbb{R}_+$  for all  $k \in \mathbb{K}$  as

$$\underline{H}_k^2(y) = \int_0^y \alpha_k^* H_k^1(\alpha_k^* x) dx \text{ and } \overline{H}_k^2(y) = \int_0^y \beta_k^* H_k^1(\beta_k^* x) dx. \quad (5)$$

We obtain (an explanation follows):

**Proposition 2.** Consider two heterogeneous distributions  $F$  and  $G$  defined over a common finite support  $[0, z]$  and lower and upper bound vectors  $\beta = (\beta_1, \dots, \beta_{K-1}, 1) \geq \alpha = (\alpha_1, \dots, \alpha_{K-1}, 1) \geq (1, \dots, 1)$ .

1.  $F \succsim_\alpha G \Leftrightarrow$

$$\sum_{i=1}^k H_i^1(z) \leq 0 \text{ for all } k \in \mathbb{K}, \text{ and} \quad (6)$$

$$\sum_{i=1}^k \underline{H}_i^2(y) \leq 0 \text{ for all } y \text{ in } [0, z] \text{ and for all } k \in \mathbb{K}. \quad (7)$$

2.  $F \succsim^\beta G \Leftrightarrow$

$$\sum_{i=k}^K H_i^1(z) \leq 0 \text{ for all } k \in \mathbb{K}, \text{ and} \quad (8)$$

$$\sum_{i=k}^K \overline{H}_i^2(y) \leq 0 \text{ for all } y \text{ in } [0, z] \text{ and for all } k \in \mathbb{K}. \quad (9)$$

**Proof.** See appendix.  $\square$

*First*, in (8) and (9), the sequential procedure is the opposite as the usual one (as in (6) and (7)), i.e., one has to check a certain condition first for type  $K$  households, then for type  $K$  and  $K - 1$  together, and so on. To put it differently, type  $K$  is considered worst-off, followed by type  $K - 1$ , and so on. This stands to reason: dividing incomes by the upper bound equivalence scale (see definition  $\overline{H}_k^2$ ), makes type  $k + 1$  households worse off compared to type  $k$  households with the same equivalent income (see assumption A2 $^\beta$ ).

*Second*, the sequential conditions (6) and (8) boil down to comparing proportions of households with certain need types between the distributions; they are due to the general way in which we deal with changing demographics as in Moyes (1999).

*Third*, the sequential dominance conditions in (7) and (9) differ from (4) in two respects, which can best be seen from the definitions of  $\underline{H}_k^2$  and  $\overline{H}_k^2$  in equation (5): the sequential dominance conditions of proposition 2 are (i) applied to equivalized incomes, either via the lower bound equivalence scales (in (7)) or via the upper bound equivalence scales (in (9)), and (ii) weighted by the equivalence scales as in Ebert (1997,1999).

*Fourth*, choosing  $\alpha = (1, \dots, 1)$ ,  $\succsim_\alpha$  equals Moyes' criterion  $\succsim$  of theorem 1. In case households only differ in household size, we could choose  $\beta^*$  equal to household size vector with a single as the reference type  $K$ . In this case, the sequential dominance conditions in (9) reduce to a *reversed* sequential dominance criterion based on per-capita household incomes and weighted by household size.

## Appendix

### Proof of proposition 1

As  $\alpha \neq \beta$ , there must exist a  $k < K$  such that  $\alpha_k \neq \beta_k$ . Evaluating equations  $A2_\alpha$  and  $A2_\beta$  for  $k$  at  $y = 0$ , we have

$$U'_k(0) = U'_{k+1}(0). \quad (10)$$

$A3_\alpha$  and (10) imply  $U'_k(\alpha_k y) \leq U'_{k+1}(y)$ , for all  $y \in \mathbb{R}_+$ ; together with  $A2_\alpha$ , we must have  $U'_k(\alpha_k y) = U'_{k+1}(y)$ , for all  $y \in \mathbb{R}_+$ . Analogously, using  $A2_\beta$ ,  $A3_\beta$  and (10) we must have  $U'_k(\beta_k y) = U'_{k+1}(y)$ , for all  $y \in \mathbb{R}_+$ . Both results together, we get

$$U'_k(\alpha_k y) = U'_k(\beta_k y), \text{ for all } y \in \mathbb{R}_+. \quad (11)$$

Equation (11) and  $\alpha_k \neq \beta_k$  are compatible if and only if

$$U'_k(y) = b, \text{ for all } y \in \mathbb{R}_+, \text{ with } b \geq 0 \text{ (from A1)}. \quad (12)$$

We now show how equation (12) extends to all other types in  $\mathbb{K} \setminus \{k\}$ .

1. Downwards. If  $k \neq 1$ , rewrite assumptions  $A2_\alpha$  and  $A2_\beta$  for  $k-1$  as

$$U'_{k-1}(y) \geq U'_k\left(\frac{y}{\alpha_{k-1}}\right) \text{ and } U'_{k-1}(y) \leq U'_k\left(\frac{y}{\beta_{k-1}}\right) \text{ for all } y \in \mathbb{R}_+. \quad (13)$$

Combining (12) and (13), we have

$$U'_{k-1}(y) = b, \text{ for all } y \in \mathbb{R}_+. \quad (14)$$

We can proceed downwards in this way until

$$U'_1(y) = \dots = U'_{k-1}(y) = b, \text{ for all } y \in \mathbb{R}_+. \quad (15)$$

2. Upwards. Use assumptions  $A2_\alpha$  and  $A2_\beta$  for  $k < K$ :

$$U'_k(\alpha_k y) \geq U'_{k+1}(y) \text{ and } U'_k(\beta_k y) \leq U'_{k+1}(y) \text{ for all } y \in \mathbb{R}_+. \quad (16)$$

Combining (12) and (16), we have

$$U'_{k+1}(y) = b, \text{ for all } y \in \mathbb{R}_+. \quad (17)$$

We can proceed upwards in this way, until

$$U'_{k+1}(y) = \dots = U'_K(y) = b, \text{ for all } y \in \mathbb{R}_+. \quad (18)$$

Integrating equations (12), (15) and (18) leads to the desired result.



## Proof of proposition 2

### Case 1: Lower bounds

**Sufficiency:** The difference in welfare between two distributions  $F$  and  $G$  equals:

$$\Delta W = \sum_{k \in \mathbb{K}} \int_0^z U_k d(p_k F_k - q_k G_k) \stackrel{a}{=} \sum_{k \in \mathbb{K}} \int_0^z U_k dH_k^1 \quad (19)$$

$$\stackrel{b}{=} \sum_{k \in \mathbb{K}} \int_0^{\alpha_k^* z} U_k dH_k^1 \stackrel{c}{=} \sum_{k \in \mathbb{K}} \int_0^z U_k(\alpha_k^* y) dH_k^1(\alpha_k^* y). \quad (20)$$

where we use (a) the definition of  $H_k^1$ , (b) the fact that  $dH_k^1$  equals zero for incomes larger than  $z$ , and (c) a change of variable. In addition, using (d and f) partial integration, (e) the definition of  $\underline{H}_k^2$  and the fact that  $H_k^1(\alpha_k^* z) = H_k^1(z)$ , we obtain

$$\Delta W \stackrel{d}{=} \sum_{k \in \mathbb{K}} U_k(\alpha_k^* z) H_k^1(\alpha_k^* z) - \sum_{k \in \mathbb{K}} \int_0^z U'_k(\alpha_k^* y) \alpha_k^* H_k^1(\alpha_k^* y) dy \quad (21)$$

$$\stackrel{e}{=} \underbrace{\sum_{k \in \mathbb{K}} U_k(\alpha_k^* z) H_k^1(z)}_A - \underbrace{\sum_{k \in \mathbb{K}} \int_0^z U'_k(\alpha_k^* y) d\underline{H}_k^2(y)}_B, \text{ with} \quad (22)$$

$$B \stackrel{f}{=} \underbrace{\sum_{k \in \mathbb{K}} U'_k(\alpha_k^* z) \underline{H}_k^2(z)}_{B1} - \underbrace{\int_0^z \left( \sum_{k \in \mathbb{K}} \alpha_k^* U''_k(\alpha_k^* y) \underline{H}_k^2(y) \right) dy}_{B2(y)}. \quad (23)$$

Using Abel's lemma,<sup>2</sup> using  $\sum_{i=1}^K \underline{H}_i^1(z) = 0$  and  $\alpha_K^* = \alpha_K = 1$ , we can rewrite  $A$ ,  $B1$ , and  $B2(y)$  as

$$\begin{aligned} A &= \sum_{k=1}^{K-1} \left[ (U_k(\alpha_k^* z) - U_{k+1}(\alpha_{k+1}^* z)) \left( \sum_{i=1}^k H_i^1(z) \right) \right], \\ B1 &= \sum_{k=1}^{K-1} \left[ (U'_k(\alpha_k^* z) - U'_{k+1}(\alpha_{k+1}^* z)) \left( \sum_{i=1}^k \underline{H}_i^2(z) \right) \right] + U'_K(z) \left( \sum_{i=1}^K \underline{H}_i^2(z) \right), \text{ and,} \\ B2(y) &= \sum_{k=1}^{K-1} \left[ (\alpha_k^* U''_k(\alpha_k^* y) - \alpha_{k+1}^* U''_{k+1}(\alpha_{k+1}^* y)) \left( \sum_{i=1}^k \underline{H}_i^2(y) \right) \right] + U''_K(y) \sum_{i=1}^K \underline{H}_i^2(y). \end{aligned}$$

As  $\Delta W = A - B1 + \int_0^z B2(y) dy$  has to be positive for all profiles satisfying assumptions A1, A2 $_{\alpha}$ , A3 $_{\alpha}$  and A4 $_{\alpha}$ , it is easy to check sufficiency of the sequential dominance conditions defined in equations (6-7).

### Necessity:

**1.** We first show that  $\sum_{i=1}^k H_k^1(z) \leq 0$  for all  $k = 1, \dots, K$  are necessary conditions. Suppose  $\Delta W \geq 0$  for all utility profiles in  $\mathcal{U}_{\alpha}$ , but there exist a  $k$  such that  $\sum_{i=1}^k H_i^1(z) > 0$ . Choose a utility profile  $(U_1, \dots, U_K)$  such that

$$[U : y \mapsto C < 0] = U_1 = \dots = U_k < U_{k+1} = \dots = U_K = [V : y \mapsto 0].$$

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<sup>2</sup>Consider two vectors  $u, v \in \mathbb{R}^K$ . Abel's lemma says that

$$\sum_{k \in \mathbb{K}} u_k v_k = \sum_{k=1}^{K-1} \left[ (u_k - u_{k+1}) \left( \sum_{i=1}^k v_i \right) \right] + u_K \sum_{i \in \mathbb{K}} v_i.$$

This profile belongs to  $\mathcal{U}_\alpha$ . The welfare difference for this profile equals

$$\Delta W = \sum_{i=1}^K \int_0^z U_i dH_i^1 = \sum_{i=1}^k \int_0^z C dH_i^1 = C \left( \sum_{i=1}^k H_i^1(z) \right) < 0,$$

a contradiction.

**2.** Suppose  $\Delta W \geq 0$  for all utility profiles in  $\mathcal{U}_\alpha$ , but there exist a  $k$  such that  $\sum_{i=1}^k \underline{H}_i^2 > 0$  on a certain non-degenerate income interval  $[a, b]$ , which belongs to  $[0, z]$ . Choose a utility profile  $(U_1, \dots, U_K)$  consisting of twice differentiable utility functions such that

$$\left\{ \begin{array}{l} U_1 : \left\{ \begin{array}{l} U_1' = C > 0 \text{ (and thus } U_1'' = 0), \text{ for } y \leq \alpha_1^* a \\ U_1'' < 0, \text{ for } \alpha_1^* a \leq y \leq \alpha_1^* b \\ U_1 = 0 \text{ (and thus } U_1' = U_1'' = 0), \text{ for } y \geq \alpha_1^* b \end{array} \right. , \\ U_2 : y \mapsto \frac{1}{\alpha_1} U_1(\alpha_1 y) = \frac{\alpha_2^*}{\alpha_1^*} U_1\left(\frac{\alpha_1^*}{\alpha_2^*} y\right), \\ U_3 : y \mapsto \frac{1}{\alpha_2} U_2(\alpha_2 y) = \frac{1}{\alpha_1 \alpha_2} U_1(\alpha_1 \alpha_2 y) = \frac{\alpha_3^*}{\alpha_1^*} U_1\left(\frac{\alpha_1^*}{\alpha_3^*} y\right), \\ \dots \\ U_k : y \mapsto \frac{\alpha_k^*}{\alpha_1^*} U_1\left(\frac{\alpha_1^*}{\alpha_k^*} y\right), \\ U_{k+1} = \dots = U_K : y \mapsto 0. \end{array} \right.$$

This profile belongs to  $\mathcal{U}_\alpha$ . The welfare difference for this profile equals

$$\begin{aligned} \Delta W &= \sum_{i=1}^K \int_0^z U_i dH_i^1 \stackrel{a}{=} \sum_{i=1}^K \int_0^{\alpha_i^* z} U_i dH_i^1 \stackrel{b}{=} \sum_{i=1}^k \int_0^{\alpha_i^* z} \frac{\alpha_i^*}{\alpha_1^*} U_1\left(\frac{\alpha_1^*}{\alpha_i^*} y\right) dH_i^1(y) \\ &\stackrel{c}{=} \sum_{i=1}^k \int_0^{\alpha_i^* b} \frac{\alpha_i^*}{\alpha_1^*} U_1\left(\frac{\alpha_1^*}{\alpha_i^*} y\right) dH_i^1(y), \end{aligned}$$

where we use (a) the fact that  $dH_k^1$  equals zero for incomes larger than  $z$ , (b) the definition of the functions in our profile, and (c) the fact that  $U_1\left(\frac{\alpha_1^*}{\alpha_i^*} y\right)$  equals zero for incomes larger than  $\alpha_i^* b$ . Furthermore, using (d and g) partial integration, (e), a change of variable, (f) the definition of  $\underline{H}_i^2$ , and (h) the fact that  $U_1''$  is zero for incomes lower than  $\alpha_1^* a$ , we obtain subsequently

$$\begin{aligned} \Delta W &\stackrel{d}{=} \underbrace{U_1(\alpha_1^* b)}_0 \left( \sum_{i=1}^k \frac{\alpha_i^*}{\alpha_1^*} H_i^1(\alpha_i^* b) \right) - \sum_{i=1}^k \int_0^{\alpha_i^* b} U_1'\left(\frac{\alpha_1^*}{\alpha_i^*} y\right) H_i^1(y) dy \\ &\stackrel{e}{=} - \sum_{i=1}^k \int_0^b U_1'(\alpha_1^* y) \alpha_i^* H_i^1(\alpha_i^* y) dy \stackrel{f}{=} - \sum_{i=1}^k \int_0^b U_1'(\alpha_1^* y) d\underline{H}_i^2(y) \\ &\stackrel{g}{=} - \underbrace{U_1'(\alpha_1^* b)}_0 \left( \sum_{i=1}^k \underline{H}_i^2(b) \right) + \int_0^b \alpha_1^* U_1''(\alpha_1^* y) \left( \sum_{i=1}^k \underline{H}_i^2(y) \right) dy \\ &\stackrel{h}{=} \int_a^b \alpha_1^* U_1''(\alpha_1^* y) \left( \sum_{i=1}^k \underline{H}_i^2(y) \right) dy, \end{aligned}$$

which is strictly negative and thus contradicts  $\Delta W \geq 0$ .

## Case 2: Upper bounds

**Sufficiency:** The difference in welfare between two distributions  $F$  and  $G$  equals:

$$\Delta W = \sum_{k \in \mathbb{K}} \int_0^z U_k d(p_k F_k - q_k G_k) \stackrel{a}{=} \sum_{k \in \mathbb{K}} \int_0^z U_k dH_k^1 \quad (24)$$

$$\stackrel{b}{=} \sum_{k \in \mathbb{K}} \int_0^{\beta_k^* z} U_k dH_k^1 \stackrel{c}{=} \sum_{k \in \mathbb{K}} \int_0^z U_k(\beta_k^* y) dH_k^1(\beta_k^* y). \quad (25)$$

steps (a) to (c) are the same as in case 1 (sufficiency). As in case (1), we proceed by using (d and f) partial integration, and (e) the definition of  $\bar{H}_k^2$  and the fact that  $H_k^1(\beta_k^* z) = H_k^1(z)$ , to get subsequently:

$$\Delta W \stackrel{d}{=} \sum_{k \in \mathbb{K}} U_k(\beta_k^* z) H_k^1(z) - \sum_{k \in \mathbb{K}} \int_0^z U'_k(\beta_k^* y) \beta_k^* H_k^1(\beta_k^* y) dy \quad (26)$$

$$\stackrel{e}{=} \underbrace{\sum_{k \in \mathbb{K}} U_k(\beta_k^* z) H_k^1(z)}_A - \underbrace{\sum_{k \in \mathbb{K}} \int_0^z U'_k(\beta_k^* y) d\bar{H}_k^2(y)}_B, \text{ with} \quad (27)$$

$$B \stackrel{f}{=} \underbrace{\sum_{k \in \mathbb{K}} U'_k(\beta_k^* z) \bar{H}_k^2(z)}_{B1} - \underbrace{\int_0^z \left( \sum_{k \in \mathbb{K}} \beta_k^* U''_k(\beta_k^* y) \bar{H}_k^2(y) \right) dy}_{B2(y)}. \quad (28)$$

Using Abel's lemma in exactly the opposite way<sup>3</sup> as well as the fact that  $\sum_{i=1}^K \bar{H}_i^1(z) = 0$  and  $\beta_K^* = \beta_K = 1$ , we can rewrite parts  $A$ ,  $B1$ , and  $B2(y)$  as

$$\begin{aligned} A &= \sum_{k=1}^{K-1} \left[ (U_{k+1}(\beta_{k+1}^* z) - U_k(\beta_k^* z)) \left( \sum_{i=k+1}^K H_i^1(z) \right) \right], \\ B1 &= \sum_{k=1}^{K-1} \left[ (U'_{k+1}(\beta_{k+1}^* z) - U'_k(\beta_k^* z)) \left( \sum_{i=k+1}^K \bar{H}_i^2(z) \right) \right] + U'_1(z) \left( \sum_{i=1}^K \bar{H}_i^2(z) \right), \text{ and,} \\ B2(y) &= \sum_{k=1}^{K-1} \left[ (\beta_{k+1}^* U''_{k+1}(\beta_{k+1}^* z) - \beta_k^* U''_k(\beta_k^* z)) \left( \sum_{i=k+1}^K \bar{H}_i^2(y) \right) \right] + U''_1(y) \sum_{i=1}^K \bar{H}_i^2(y). \end{aligned}$$

As  $\Delta W = A - B1 + \int_0^z B2(y) dy$  has to be positive for all profiles satisfying assumptions A1, A2 <sup>$\beta$</sup> , A3 <sup>$\beta$</sup> , and A4 <sup>$\beta$</sup> , it is easy to check sufficiency of the sequential dominance conditions defined in equations (8-9).

### Necessity:

**1.** We first show that  $\sum_{i=k}^K H_i^1(z) \leq 0$  for all  $k = 1, \dots, K$  are necessary conditions. Suppose  $\Delta W \geq 0$  for all utility profiles in  $\mathcal{U}^\beta$ , but there exist a  $k$  such that  $\sum_{i=k}^K H_i^1(z) > 0$ . Choose a utility profile  $(U_1, \dots, U_K)$  with

$$[V : y \mapsto 0] = U_1 = \dots = U_{k-1} > U_k = \dots = U_K = [U : y \mapsto C < 0].$$

<sup>3</sup>Consider two vectors  $u, v \in \mathbb{R}^K$ . Here we use Abel's lemma as follows:

$$\sum_{k \in \mathbb{K}} u_k v_k = \sum_{k=1}^{K-1} \left[ (u_{k+1} - u_k) \left( \sum_{i=k+1}^K v_i \right) \right] + u_1 \sum_{i \in \mathbb{K}} v_i.$$

For this profile, which belongs to  $\mathcal{U}^\beta$ , the welfare difference can be rewritten as

$$\Delta W = \sum_{i=1}^K \int_0^z U_i dH_i^1 = \sum_{i=k}^K \int_0^z C dH_i^1 = C \left( \sum_{i=k}^K H_i^1(z) \right) < 0,$$

a contradiction.

**2.** Suppose  $\Delta W \geq 0$  for all utility profiles in  $\mathcal{U}^\beta$ , but there exist a  $k$  such that  $\sum_{i=k}^K \overline{H}_i^2 > 0$  on a certain non-degenerate income interval  $[a, b]$ , which belongs to  $[0, z]$ . Choose a utility profile  $(U_1, \dots, U_K)$  parametrized by  $\alpha > 0$  in the following way:

$$\left\{ \begin{array}{l} U_K : y \mapsto \begin{cases} (2\alpha(b-a) + 1)y, & \text{for } y \leq a \\ -\alpha y^2 + (2\alpha b + 1)y - \alpha a^2, & \text{for } a \leq y \leq b \\ y + \alpha(b^2 - a^2), & \text{for } y \geq b \end{cases} , \\ U_{K-1} : y \mapsto \beta_{K-1} U_K \left( \frac{y}{\beta_{K-1}} \right) = \beta_{K-1}^* U_K \left( \frac{y}{\beta_{K-1}^*} \right), \\ U_{K-2} : y \mapsto \beta_{K-2} U_{K-1} \left( \frac{y}{\beta_{K-2}} \right) = \beta_{K-2}^* U_K \left( \frac{y}{\beta_{K-2}^*} \right), \\ \dots \\ U_k : y \mapsto \beta_k U_{k+1} \left( \frac{y}{\beta_k} \right) = \dots = \beta_k^* U_K \left( \frac{y}{\beta_k^*} \right), \\ U_{k-1} = \dots = U_1 : y \mapsto y + \beta_k^* \alpha (b^2 - a^2). \end{array} \right.$$

The function  $U_K$ —and thus also the functions  $U_{K-1}, \dots, U_k$ —as well as the functions  $U_{k-1} = \dots = U_1$  are twice differentiable and positively valued. Moreover, this profile belongs to  $\mathcal{U}^\beta$ , irrespective of  $\alpha > 0$ , which defines the degree of concavity of  $U_K$ . Independent of the value of  $\alpha$ , there exists an income vector  $(z_1, \dots, z_K)$  with all  $z_k \geq \max(z, \beta_k^* b)$  such that

$$U_k(z_k) = \overline{U}, \text{ for all } k \in \mathbb{K} \text{ and } U'_k(z_k) = 1, \text{ for all } k \in \mathbb{K}.$$

The welfare difference for this profile equals

$$\begin{aligned} \Delta W &= \sum_{i=1}^K \int_0^z U_i dH_i^1 \stackrel{a}{=} \sum_{i=1}^K \int_0^{z_i} U_i dH_i^1 \stackrel{b}{=} \sum_{i=1}^K U_i(z_i) H_i^1(z_i) - \sum_{i=1}^K \int_0^{z_i} U'_i H_i^1 \\ &\stackrel{c}{=} - \sum_{i=1}^K \int_0^{z_i} U'_i H_i^1 \stackrel{d}{=} - \sum_{i=1}^K \int_0^{\frac{z_i}{\beta_i^*}} U'_i(\beta_i^* y) H_i^1(\beta_i^* y) d(\beta_i^* y) \\ &\stackrel{e}{=} - \sum_{i=1}^K \int_0^{\frac{z_i}{\beta_i^*}} U'_i(\beta_i^* y) d\overline{H}_i^2(y) \\ &\stackrel{f}{=} - \sum_{i=1}^K \overline{H}_i^2 \left( \frac{z_i}{\beta_i^*} \right) + \sum_{i=k}^K \int_a^b \beta_i^* U''_i(\beta_i^* y) \overline{H}_i^2(y) dy \\ &\stackrel{g}{=} - \sum_{i=1}^K \overline{H}_i^2 \left( \frac{z_i}{\beta_i^*} \right) - \alpha \int_a^b \left( \sum_{i=k}^K \overline{H}_i^2(y) \right) dy, \end{aligned}$$

where we use (a) the fact that  $dH_k^1$  equals zero above  $z$ , (b) partial integration, (c) the fact that  $U_i(z_i) = \overline{U}$  and  $\sum_{i=1}^K H_i^1(z_i) = \sum_{i=1}^K H_i^1(z) = 0$ , (d) a change of variable, (e)

the definition of  $\overline{H}_k^2$ , (f) partial integration,  $U_i'(z_i) = 1$  for all  $i \in \mathbb{K}$  and the fact that —for all  $i = 1, \dots, k-1$ —  $U_i''$  equals zero and —for all other  $i = k, \dots, K$ —  $U_i''$  equals zero for incomes lower (resp. higher) than  $\beta_i^*a$  (resp.  $\beta_i^*b$ ), and (g) the fact that —for all  $i = k, \dots, K-1$ —  $\beta_i^*U_i''(\beta_i^*y) = U_K''(y) = -\alpha$  for incomes  $y$  in  $[a, b]$ . The right-hand side of (g) can be made arbitrarily low by increasing  $\alpha$ , thus strictly negative as well, which contradicts  $\Delta W \geq 0$ .

## References

- [1] Atkinson, A.B. and Bourguignon, F. (1987), Income distribution and differences in needs, in G.R. Feiwel (ed.), *Arrow and the Foundations of the Theory of Economic Policy*, London:Macmillan.
- [2] Atkinson, A.B. (1992), Measuring poverty and differences in family composition, *Economica* 59, 1-16.
- [3] Bourguignon, F. (1989), Family size and social utility, Income distribution dominance criteria, *Journal of Econometrics* 42, 67-80.
- [4] Chambaz, E. and Maurin, C. (1998), Atkinson and Bourguignon's dominance criteria: extended and applied to the measurement of poverty in France, *Review of Income and Wealth* 44, 497-513.
- [5] Cowell, F.A. and Mercader-Prats, (1999), *Equivalence Scales and Inequality*, in J. Silber (ed.), Income inequality measurement: from theory to practice, Denter: Kluwer.
- [6] Ebert, U. (1997), Social welfare when needs differ: An axiomatic approach, *Economica* 64, 233-244.
- [7] Ebert, U. (1999), Using equivalent income of equivalent adults to rank income distributions, *Social Choice and Welfare* 16, 233-258.
- [8] Ebert, U. (2000) Sequential generalized Lorenz dominance and transfer principles, *Bulletin of Economic Research* 52 (2), 113-122.
- [9] Fleurbaey, M., Hagneré, C. and Trannoy, A. (2001), Welfare comparisons with bounded equivalence scales, *Journal of Economic Theory*, forthcoming.
- [10] Jenkins, S.P. and Lambert, P.J. (1993), Ranking income distributions when needs differ, *Review of Income and Wealth* 39, 337-356.
- [11] Lambert, P.J. (2001), *The Distribution and Redistribution of Income. A Mathematical Analysis*, 3rd edition, Manchester: Manchester University Press.
- [12] Lambert, P.J. and Ramos, X. (2002), Welfare comparisons: sequential procedures for heterogeneous populations, *Economica* 69, 549-562.
- [13] Moyes, P. (1999), Comparaisons de distributions hétérogènes et critères de dominance, *Economie et Prévision* 138-139, 125-146.

- [14] Shorrocks, A.F. (1995), Inequality and welfare evaluation of heterogeneous income distributions, Discussion Paper 447, University of Essex.

